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THEORETICAL ANALYSIS OF THE STATIC GENERAL INSTABILITY OF AN ORTHOTROPIC CIRCULAR CYLINDER SUBJECTED TO AN AXIAL LOAD, END MOMENT AND UNIFORM RADIAL PRESSURE

UNPUBLISHED PRELIMINARY DATA

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by

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I. SCOPE

This analysis obtains numerical results using a digital computer program for the general instability eigenvalue problem that is presented for the dependent buckling load condition at any combination of the independent loading and geometry. It is not intended that this analysis be experimentally verified as a part of this investigation.

II. INTRODUCTION

Missile tank design is subjected to two design criteria; the material strength for all possible maximum load conditions, and the structural stability at these possible maximum load conditions as well as intermediate loads. The analysis of this report is limited to the stability criteria of missile design.

A missile tank is loaded with different combinations of axial load, end moment, and radial pressure. The axial load, either compressive or tensile, is a constant force per unit cross-sectional area and is colinear with the generating element of the tank. The end moment is a varying force per unit cross-sectional area and is colinear with the generating element of the tank. This force varies linearly with the distance between a diameter that is normal to the plane of the moment, and the element of cross-sectional area. The radial pressure is internal or external, depending upon the sign given to the pressure difference.

The positive directions of axial load, end moment, and radial pressure are as shown in Figure 2 and are chosen to induce tension on the element of the tank at the origin of the axes, Figure 1.

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Functions performed by missile tanks require that they be stiffened axially and circumferentially with internal baffles, internal and external stiffeners, bulkheads, etc. These baffles and stiffeners are integral parts of the missile tank and it becomes possible to analyze the tank as an orthotropic circular cylindrical shell. The orthotropic circular cylindrical shell is called circular shell or shell in the remainder of this report.

The loads applied to the shell are not functions of time, therefore, the investigation is limited to the static case, and the dynamic case is neglected.

The technique used in this investigation of the cylindrical shell parallels the work of Bodner (1)*, in that the general instability differential equation of equilibrium developed is a Donnell type differential equation and is obtained by the application of variational methods to the expression for total change in energy during buckling.

Results can be obtained from the Donnell Type differential equation by any one of several different methods: Ritz Method, Fourier Series Method, Galerkin Method, Method of Frobenius, etc.; all of which will yield a satisfactory solution. The Ritz Method is used in this investigation. The results obtained by the Ritz Method are as accurate as the assumed deflection expression and the solution becomes an exact solution when the deflection expression takes the form of an infinite Fourier series. The Ritz Method is mathematically the simplest of the methods mentioned above and it is readily programmed for digital computer applications.

The points of stability for the total change in energy expression of a conservative system are defined by the law of minimum potential energy when the variational principal is applied to the total change in energy. The first variation of the total change in energy is equated to zero, thereby obtaining the intrinsic boundary conditions and the equilibrium equations of the system. The Donnell Type differential equation is obtained by applying a differential operator to the equilibrium equations of the system. An assumed deflection expression is substituted into the Donnell equation and the resulting residual force equation is minimized for each of the unknown constants of the deflection expression. This minimization yields a system of homogeneous simultaneous equations, and the stability determinant of these equations is solved for the independent variable.

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*Indicates reference number - see Appendix B.

The law of minimum potential energy requires that the second variation of the total change in energy expression of the system be positive for the points of stable equilibrium and negative for points of unstable equilibrium. The second variation is not performed due to the anticipated difficulty of the mathematics, and the minimum positive value of the independent variable is assumed as the point of stable equilibrium.

Experimental evidence obtained by Harris, Suer, Skene and Benjamin (2) indicates that isotropic circular shell test specimens subjected to axial load with and without radial pressure fail somewhere between the stable and unstable equilibrium points and that the failure point is primarily dependent upon the quality of the specimen. The almost perfect specimens fail at the point approaching the point of unstable equilibrium. As the imperfections of the specimens become larger or more numerous the specimen fails at a point closer to the point of stable equilibrium. Theories developed for unpressurized cylinders with axial loads by von Kármán and Tsien (3), Leggett and Jones (4), and Tsien (5) using the large deflection theory have attempted to explain the deviation between theoretical and experimental results. These theories are still considered as inadequate since their results cannot be readily adapted as design criteria. A similar approach was used for pressurized cylinders with axial loads by Donnell and Wan (6) with more success, but a deviation still exists.

Small deflection theory of shell analysis states that all terms greater than second degree in the total change in energy expression may be neglected. The small deflection theory is used in this investigation and allows the development of the Donnell Type linear differential equation which can be readily solved for a certain particular type of loading. That the small deflection theory is applicable to certain shell configurations is questioned by some investigators as indicated above. The answer to this question is left for further analytical work associated with the experimental evaluation of this investigation.

The classical small deflection theory for isotropic shell stability is limited to the range R/h-values** less than 200. Some missile tanks have R/h-values of 1000 and there have been indications that this value may reach 2000. This indicates that the R/h-values for orthotropic shells that represent stiffened shells should be modified, or the valid range

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** See Appendix A for list of symbols and definitions.

of the theory extended for orthotropic shells. Arguments for a modification of R/h-values using a modified h-value, which we will call (h_{eq}), are based on the dependence of the stability criterion on the bending rigidity of the shell. Similar arguments are used for the extensional stiffness. Suggested values for (h_{eq}) are:

$$h_{eq} = (12 \ I_{eq})^{1/3}$$

where I_{eq} is the composite moment of inertia of the shell plus stiffeners;

$$h_{eq} = [6(I_{xx} + I_{ss})]^{1/3}$$

where I_{xx} and I_{ss} are the equivalent composite moments of inertia in the axial and circumferential directions, respectively; or $h_{eq} = f(r)$, where r is the radius of gyration of a unit element of the orthotropic shell. It is believed that if the shell is analyzed with $R/h_{eq} = 200$ the small deflection theory will be applicable.

Investigation is being conducted (7) which may allow the proper selection of an R/h-value for orthotropic shells with an h_{eq}-value. Again the question of an R/h_{eq}-value for orthotropic shells is left for experimental evaluation and/or results of investigations in progress.

For short cylinders $(R \geq L)$ the assumed deflection expression, Equation 26, reduces to the Euler column expression when the cylinder is simply supported, if the circumferential deflection terms become constant. For long cylinders $(R \leq L/3)$ the buckling becomes independent of the boundary conditions. In these ranges this analysis is valid for values of $\pi R/L$, but the intermediate range $(L \geq R \geq L/3)$ the results should again be experimentally verified.

III. ASSUMPTIONS

The following assumptions are made in this analysis of circular shells.

1. The shell is composed of linearly elastic material.

2. The stiffeners and baffles are integral parts of the shell, thus creating an orthotropic shell, and the unstiffened and unbaffled shell reduces to the isotropic case.

3. The shell stresses in the unbuckled but stressed state are determined by elementary beam theory.

4. The strain equations, Equation 1, are similar to those used by Bodner (1) except for certain second degree terms. The second degree terms are included in this analysis since their effect although unknown, is considered significant. The last term on the right hand side of the e_{ss} equation, Equation 1, includes a non-dimensional constant k. With values of k equal to one and zero, the effect of this term on the final results can be determined.

5. The work in the circumferential direction is neglected in the determination of the total change in energy expression. This is based on the symmetry of both of the σ_{ss} stresses and the cylindrical shell geometry.

6. The pre-buckling deformation discussed by Donnell and Wan (6) and Stein (8) are neglected. The effect of these deformations should be investigated during experimental verification.

7. In the development of the Donnell equation, all terms above the second order in the total energy expression are discarded. Neglecting the terms above second order simplifies the mathematics and insures a small deflection theory approach to the analysis.

8. Localized or panel instability is neglected in this analysis and only the general instability of the shell is investigated.

9. The assumed deflection expression, Equation 26, contains only 6 circumferential terms, but can be extended to any number desired. The use of six terms requires that a cubic equation be solved and this solution can easily be programmed. Additional terms in the deflection expression would complicate the computer program and possibly produce a computer overload. The axial term in the deflection expression contains a nondimensional constant m which is used to obtain any number of buckling modes.

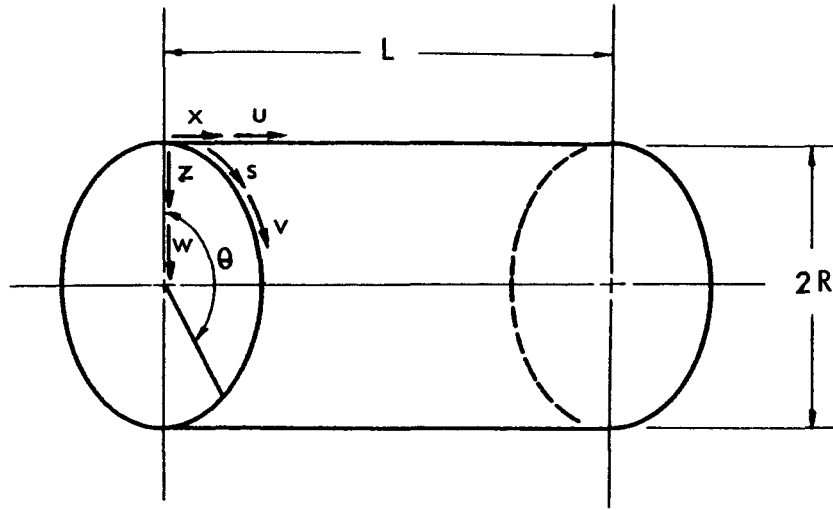## IV. CYLINDRICAL SHELL GEOMETRY AND STRESS-STRAIN RELATIONS

Fig. 1: Coordinate System and Displacements of the Circular Cylindrical Shell.

The coordinate system and corresponding middle-surface displacements for the circular cylindrical shell are shown in Figure 1.

The expressions used for the buckling strains in the shell wall; written in terms of the shell middle-surface displacements, u, v, and w; are the same as those given in Reference (1) with some additional terms, and are written as follows:

$$e_{xx} = u_{,x} + (1/2) w^2_{,x} - z w_{,xx}$$

$$e_{ss} = v_{,s} - (w/R) + (1/2)(w_{,s} + v/R)^2 - z(w_{,ss} + kw/R^2) \tag{1}$$

$$e_{xs} = (1/2) [u_{,s} + v_{,x} + w_{,x}(w_{,s} + v/R)] - z(w_{,xs} + v_{,x}/2R)$$

where e_{xx}, e_{ss}, and e_{xs}, are the axial, circumferential, and shear strains, respectively, that occur during the buckling process; R is the radius of the cylinder; and k is a nondimensional constant. When the subscript or subscripts associated with the middle-surface displacements are preceeded by a comma, they denote differentiation with respect to the indicated succeeding coordinate variables.

The stress-strain relationships for a homogenous orthotropic material in generalized

plane stress, as given by Reference (1), can be written as follows:

$$\sigma_{xx} = E_x (e_{xx} + \mu_{sx} e_{ss})/(1 - \mu_{sx} \mu_{xs})$$

$$\sigma_{ss} = E_s (e_{ss} + \mu_{xs} e_{xx})/(1 - \mu_{sx} \mu_{xs}) \tag{2}$$

$$\sigma_{xs} = G e_{xs}$$

where σ_{xx}, σ_{ss}, and σ_{xs}, are the axial, circumferential, and shear stresses, respectively; E_x and E_s are the moduli of elasticity averaged over the shell thickness in the axial and circumferential directions, respectively; G is the average shear modulus; and μ_{xs} and μ_{sx} are Poisson's ratios from the x to the s and s to the x directions, respectively.

For convenience in later calculations certain constants, similar to those given in Reference (1), are introduced and are written as follows:

$$\alpha_1 = E_x h /2(1 - \mu_{xs}\mu_{sx})$$

$$\alpha_2 = E_s h /2(1 - \mu_{xs}\mu_{sx})$$

$$\alpha_3 = Gh/8 \tag{3}$$

$$D_1 = E_x h^3/24(1 - \mu_{xs}\mu_{sx})$$

$$D_2 = E_s h^3/24(1 - \mu_{xs}\mu_{sx})$$

$$D_3 = Gh^3/96$$

where h is the shell thickness; the α's correspond to the extensional stiffness of the shell; and the D's correspond to the bending rigidity of the shell.

The following relationship between the elastic constants, based on Maxwell's reciprocal theorem, is noted for later use.

$$E_s \mu_{xs} = E_x \mu_{sx} \tag{4}$$
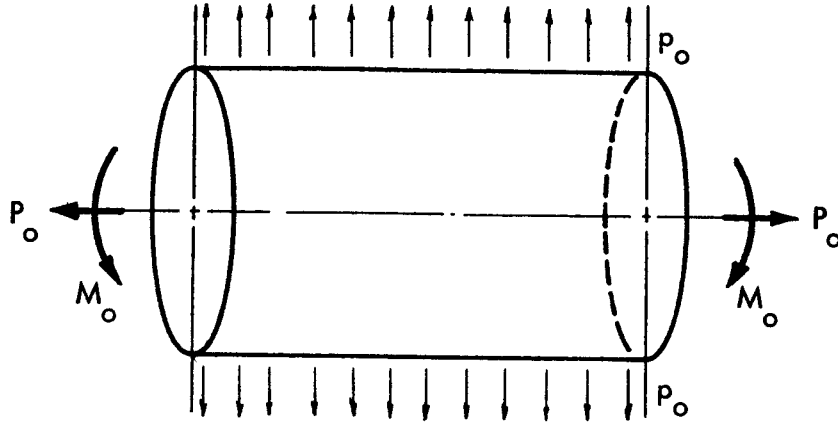
V. CYLINDRICAL SHELL LOADING AND STRESS RESULTANTS

Fig. 2: Loading of the Circular Cylindrical Shell

The positive shell loads p_o, P_o, and M_o are shown with respect to the coordinate system; where p_o is the uniform radial pressure, P_o is the resultant axial load, and M_o is the resultant end moment. The positive loads are directed in order that positive stresses (tensil) are induced at the origin of the coordinate system.

The following stress resultants are defined:

$$\overline{N}_{xx} = \int_{-h/2}^{h/2} \overline{\sigma}_{xx} dz \qquad \overline{N}_{ss} = \int_{-h/2}^{h/2} \overline{\sigma}_{ss} dz \qquad \overline{N}_{xs} = \int_{-h/2}^{h/2} \overline{\sigma}_{xs} dz \qquad (5)$$

where $\overline{N}_{xx}$, $\overline{N}_{ss}$, and $\overline{N}_{xs}$ are the axial, circumferential, and shear stress resultants in the shell wall, respectively, prior to buckling; and $\overline{\sigma}_{xx}$, $\overline{\sigma}_{ss}$, and $\overline{\sigma}_{xs}$ are the axial, circumferential, and shear stresses in the shell wall, respectively, prior to buckling. In general the barred symbols indicate stresses, strains, and stress resultants in the shell prior to buckling, while un-barred symbols indicate stresses and strains that occur in the shell during the buckling process.

According to elementary beam and shell theory, the shell loading will induce the following stresses in the shell wall.

$$\bar{\sigma}_{xx} = (1/h) \, [M_a \cos(s/R) + (R/2) \, (P_a + p_o)]$$

$$\bar{\sigma}_{ss} = (1/h) \, p_o R \tag{6}$$

$$\bar{\sigma}_{xs} = 0$$

where $M_a = M_o/\pi R^2$ and $P_a = P_o/\pi R^2$.

Substituting Equation (6) into Equation (5) and integrating over the shell thickness the stress resultants become:

$$\overline{N}_{ss} = p_o R$$

$$\overline{N}_{xx} = M_a \cos (s/R) + (R/2)(p_o + P_a) \tag{7}$$

VI. STRAIN ENERGY, POTENTIAL ENERGY, TOTAL CHANGE IN ENERGY, AND VARIATION IN TOTAL CHANGE IN ENERGY EXPRESSIONS.

The instability differential equations of equilibrium will be derived using a procedure similar to that given in Reference (1). The criterion of buckling for an elastic system is that the potential energy of the system is a minimum. Stated mathematically, the variation of the change in energy of the system due to buckling, with respect to the displacements, must be zero; or:

$$\delta \, (U + V) = 0 \tag{8}$$

where U is the change in strain energy of the shell during buckling, V is the change in potential energy of the applied loads during buckling, and δ indicates a variation of the sum with respect to displacements.

The change in the strain energy of the shell is given by the following expression:

$$U = \int_{V_s} [(\bar{\sigma}_{xx} e_{xx} + \bar{\sigma}_{ss} e_{ss} + \bar{\sigma}_{xs} e_{xs}) + (1/2)(\sigma_{xx} e_{xx} + \sigma_{ss} e_{ss} + \sigma_{xs} e_{xs})] \, dV_s \tag{9}$$

where $\bar{\sigma}_{xx}$, $\bar{\sigma}_{ss}$, and $\bar{\sigma}_{xs}$ are the membrane stresses in the shell wall in the stressed but unbuckled state and they are assumed to be constant during buckling; σ_{xx}, σ_{ss}, and σ_{xs} are

the superimposed buckling stresses; e_{xx}, e_{ss}, and e_{xs} are the buckling strains; and V_s is the volume of the shell wall.

The change in potential energy of the shell during buckling has components in the z and x directions and the total change in potential energy during buckling is given by the following expression:

$$V = - \int_{A_s} P_o(-w)dA_s - \int_{A_e} \bar{\sigma}_{xx} u \, dA_e = \int_{A_s} P_o w dA_s - \int_{A_s} h \bar{\sigma}_{xx} u_{,x} dA_s \tag{10}$$

where A_s and A_e are the surface and cross sectional areas of the shell, respectively; and u is given by the expression $u = \int_o^L u_{,x} dx$.

The total change in energy, strain energy plus potential energy, during buckling is obtained by adding Equation (9) to Equation (10); substituting Equations (1), (2), (4), (5), (6), and (7) into Equations (9) and (10); and integrating over the shell thickness. The expression for total change in energy is given by the following expression:

$$
\begin{aligned}
U + V = \int_{A_s} \{ & P_o [v_{,s} R + (w_{,s}^2 R/2) + w_{,s} v + (v^2/2R) + (w_{,x}^2 R/4)] \\
& + P_a [w_{,x}^2 R/4] + M_a \cos(s/R)[w_{,x}^2/2] \} \, dA_s \\
& + \int_{A_s} \{ [E_x h/2(1-\mu_{xs}\mu_{sx})][u_{,x}^2] + [E_s h/2(1-\mu_{xs}\mu_{sx})][v_{,s}^2 + (w^2/R^2) \\
& \quad - (2v_{,s}w/R) + \mu_{xs} u_{,x} v_{,s} - (\mu_{xs} u_{,x} w/R)] \\
& + [Gh/8][v_{,x}^2 + u_{,s}^2 + w_{,x}^2 w_{,s}^2 + (w_{,x}^2 v^2/R^2) + 2v_{,x} u_{,s} + 2v_{,x} w_{,x} w_{,s} \\
& \quad + (2v_{,x} w_{,x} v/R) + 2u_{,s} w_{,x} w_{,s} + (2u_{,s} w_{,x} v/R) + (2w_{,x}^2 w_{,s} v/R)] \} \, dA_s \\
& + \int_{A_s} \{ [h^3 E_x/24(1-\mu_{sx}\mu_{xs})][w_{,xx}^2 + \mu_{sx} w_{,xx} w_{,ss} + (k \mu_{sx} w_{,xx} w/R^2)] \\
& + [h^3 E_s/24(1-\mu_{sx}\mu_{xs})][w_{,ss}^2 + (2w_{,ss} kw/R^2) + (k^2 w^2/R^4) \\
& \quad + \mu_{xs} w_{,xx} w_{,ss} + (\mu_{xs} kw_{,xx} w/R^2)] \\
& + [Gh^3/96][4w_{,xs}^2 + (v_{,x}^2/R^2) + (4w_{,xs} v_{,x}/R)] \} dA_s
\end{aligned}
\tag{11}
$$

Substituting Equation (3) into Equation (11) and discarding all third order and greater terms the expression for total change in energy reduces to:

$$U + V = \int_{A_s} \{ P_o [v,_s R + (w,_s^2 R/2) + w,_s v + (v^2/2R) + (w,_x^2 R/4)]$$

$$+ P_a [w,_x^2 R/4] + M_a \cos(s/R) [w,_x^2/2]$$

$$+ a_1 [u,_x^2] + a_2 [v,_s^2 + (w^2/R^2) - (2v,_s w/R) + \mu_{xs} u,_x v,_s - (\mu_{xs} u,_x w/R)]$$

$$+ a_3 [v,_x^2 + u,_s^2 + 2v,_x u,_s] + D_1 [w,_{xx}^2 + \mu_{sx} w,_{xx} w,_{ss} + (k \mu_{sx} w,_{xx} w/R^2)] \qquad (12)$$

$$+ D_2 [w,_{ss}^2 + (2w,_{ss} kw/R^2) + (k^2 w^2/R^4) + \mu_{xs} w,_{xx} w,_{ss} + (\mu_{xs} kw,_{xx} w/R^2)]$$

$$+ D_3 [4w,_{xs}^2 + (v,_x^2/R^2) + (4w,_{xs} v,_x/R)] \} dA_s$$

where a_1, a_2, a_3, D_1, D_2, and D_3 are as defined in Equation (3). Several authors, including the author of Reference (1) have proven that the ommitted terms are negligible. The use of terms up to the second order will result in a linear differential equation.

Applying the variational principal,

$$\delta F = \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' + \frac{\partial F}{\partial y''} \delta y'' + \ldots \ldots \ldots + \frac{\partial F}{\partial y^n} \delta y^n, \qquad (13)$$

to Equation (12), the following expression for the variation in the total change in energy with respect to the displacements u, v, and w is obtained:

$$\delta(U+V) = \int_{A_s} \Bigg[\{ a_2 [(2w/R^2) - (2v,_s/R) - (\mu_{xs} u,_x/R)]$$

$$+ D_2 [(2w,_{ss} k/R^2) + (2\mu_{xs} w,_{xx} k/R^2) + (2k^2 w/R^4)] \} \delta w$$

$$+ \{ M_a w,_x \cos(s/R) + (p_o Rw,_x/2) + (P_a Rw,_x/2) \} \delta w,_x \qquad (14)$$

$$+ \{ p_o (w,_s R + v) \} \delta w,_s$$

$$+ \{ D_1 [2w,_{xx} + 2\mu_{sx} w,_{ss} + (2\mu_{sx} kw/R^2)] \} \delta w,_{xx}$$

$$+ \{D_2 [2w,_{ss} + (2kw/R^2) + 2\mu_{xs}w,_{xx}]\} \delta w,_{ss}$$

$$+ \{D_3 [8w,_{xs} + (4v,_x/R)]\} \delta w,_{xs}$$

$$+ \{a_1 [2u,_x] + a_2 [\mu_{xs}v,_s - (\mu_{xs}w/R)]\} \delta u,_x$$

$$+ \{a_3 [2u,_s + 2v,_x]\} \delta u,_s + \{p_o [(v/R) + w,_s]\} \delta v$$

$$+ \{a_3 [2v,_x + 2v,_s] + D_3 [(2v,_x/R^2) + (4w,_{xs}/R)]\} \delta v,_x$$

$$+ \{p_o R + a_2 [2v,_s - (2w/R) + \mu_{xs}u,_x]\} \delta v,_s \bigg] dA_s$$

(14 Con't.)

Equation (14) is simplified further by setting $dA_s = dxds$; applying the following identity from calculus of variations, $\delta \dfrac{dx}{dy} = \dfrac{d}{dy}(\delta_x)$; and integrating between the limits of 0 to L for dx, and 0 to $2\pi R$ for ds. The final form of the expression for the variation in the total change in energy of the orthotropic cylindrical shell during the buckling process is as follows:

$$\delta(U + V) = \int_0^L \int_0^{2\pi R} \bigg[\{2a_2 [(w/R^2) - (v,_s/R) - (\mu_{xs}u,_x/2R)]$$

$$+ 2D_2 [(2w,_{ss}k/R^2) + (\mu_{xs}w,_{xx}k/R^2) + (k^2 w/R^4) + w,_{ssss} + \mu_{xs}w,_{xxss}]$$

$$- [M_a w,_{xx}\cos(s/R) + (Rw,_{xx}/2)(p_o + P_a)] - [p_o(w,_{ss}R + v,_s)]$$

$$+ 2D_1 [w,_{xxxx} + \mu_{sx}w,_{xxss} + (\mu_{sx}kw,_{xx}/R^2)]$$

$$+ 4D_3 [2w,_{xxss} + (v,_{xxs}/R)]\} \delta w$$

$$+ \{-2 a_1 [u,_{xx} + (\mu_{sx}v,_{xs}/2) - (\mu_{sx}w,_x/2R)] - 2 a_3 [u,_{ss} + v,_{xs}]\} \delta u$$

$$+ \{-2a_2 [v,_{ss} - (w,_s/R) + (\mu_{xs}u,_{xs}/2)] + [p_o(w,_s + \tfrac{v}{R})]$$

$$- 2 a_3 [v,_{xx} + u,_{xs}] - 4D_3 [(v,_{xx}/2R^2) + (w,_{xxs}/R)]\} \delta v \bigg] dxds$$

(15)

$$+ \int_0^L \left[\{[p_o(w,_s R + v)] - 2D_2[w,_{sss} + \mu_{xs} w,_{xxs} + (kw,_s/R^2)] \right.$$

$$\left. -4D_3[2w,_{xxs} + (v,_{xx}/R)]\}\Big|_o^{2\pi R} \, \delta w \right.$$

$$+\{2a_3[u,_s + v,_x]\}\Big|_o^{2\pi R} \, \delta u$$

$$+\{p_o R + 2a_2[v,_s -(w/R) + (\mu_{xs} u,_x/2)]\}\Big|_o^{2\pi R} \, \delta v$$

$$\left. +\{2D_2[w,_{ss} + \mu_{xs} w,_{xx} + (kw/R^2)]\}\Big|_o^{2\pi R} \, \delta w,_s \right] \, dx$$

$$+ \int_0^{2\pi R} \left[\{[m_a w,_x \cos(s/R) + (Rw,_x/2)(p_o + P_a)] \right.$$

$$-2D_1[w,_{xxx} + \mu_{sx} w,_{xss} + (\mu_{sx} k w,_x/R^2)] \qquad \text{(15 Con't.)}$$

$$\left. -4D_3[2w,_{xss} + (v,_{xs}/R)]\}\Big|_o^L \, \delta w \right.$$

$$+\{2a_1[u,_x] + 2a_2[(\mu_{xs} v,_s/2) - (\mu_{xs} w/2R)]\}\Big|_o^L \, \delta u$$

$$+\{2a_3[v,_x + u,_s] + 4D_3[(v,_x/2R^2) + (w,_{xs}/R)]\}\Big|_o^L \, \delta v$$

$$\left. +\{2D_1[w,_{xx} + \mu_{sx} w,_{ss} + (\mu_{sx} k w/R^2)]\}\Big|_o^L \, \delta w,_x \right] \, ds$$

$$+ \left[\{4D_3[2w,_{xs} + (v,_x/R)]\}\Big|_o^L \right]\Big|_o^{2\pi R} \, \delta w$$

VII. EQUILIBRIUM EQUATIONS AND NATURAL BOUNDARY CONDITIONS

The variation in the total change in energy of the system must vanish for any of the arbitrary virtual displacements δu, δv, and δw when the system is in equilibrium. When Equation (15) is equated to zero the integrands of the surface integral must vanish, since the virtual displacements are arbitrary, and the following stability equilibrium equations are obtained:

$$u,_{xs}[-a_3 -(a_2 \mu_{xs}/2)] + v[p_o/2R] + v,_{xx}[(-D_3/R^2) - a_3] + v,_{ss}[-a_2]$$

$$+ w,_{xxs}[-2D_3/R] + w,_s[(p_o/2) - (a_2/R)] = 0 \qquad (16)$$

$$v,_{xs} [-\alpha_3-(\alpha_2 \mu_{xs}/2)] + u,_{xx} [-\alpha_1] + u,_{ss} [-\alpha_3] + w,_x [\alpha_2 \mu_{xs}/2R] = 0 \qquad (17)$$

$$u,_x [-\alpha_2 \mu_{xs}/2R] + v,_s [(-\alpha_2/R) + (-p_o/2)] + v,_{xxs} [2D_3/R]$$

$$+ w [(\alpha_2/R^2) + (k^2 D_2/R^4)] + w,_{xx} [(2D_1 \mu_{sx} k/R^2) - (m_a/2) \cos (s/R) - (R/4)(p_o + P_a)] \qquad (18)$$

$$+ w,_{ss} [(2D_2 k/R^2) - (p_o R/2)] + w,_{xxxx} [D_1] + w,_{ssss} [D_2]$$

$$+ w,_{xxss} [2\mu_{sx} D_1 + 4D_3] = 0$$

The following natural boundary conditions are obtained from Equation (15) when the constant term and the integrands of the line integrals vanish for any arbitrary virtual displacement, and derivative of an arbitrary virtual displacement.

$$\left[[w,_{xs} + (v,_x/2R)]_0^L \right]_0^{2\pi R} = 0$$

$$[w,_{xx} + \mu_{sx} w,_{ss} + (k \mu_{sx} w/R^2)]_0^L = 0$$

$$[w,_{ss} + \mu_{xs} w,_{xx} + kw/R^2]_0^{2\pi R} = 0$$

$$[u,_s + v,_x]_0^{2\pi R} = 0$$

$$[2u,_x + \mu_{sx} v,_s - (\mu_{sx} w/R)]_0^L = 0$$

$$[p_o R + 2\alpha_2 (v,_s - w/R + \mu_{xs} u,_x/2)]_0^{2\pi R} = 0 \qquad (19)$$

$$[\alpha_3(v,_x + u,_s) + 2D_3 (v,_x/2R^2 + w,_{xs}/R)]_0^L = 0$$

$$[p_o(w,_s R + v) - 2D_2 (w,_{sss} + \mu_{xs} w,_{xxs} + kw,_s/R^2)$$

$$-4D_3(2w,_{xxs} + v,_{xx}/R)]_0^{2\pi R} = 0$$

$$[M_a w,_x \cos (s/R) + (Rw,_x/2)(p_o + P_a) - 2D_1 (w,_{xxx} + \mu_{sx} w,_{xss} + \mu_{sx} kw,_x/R^2)$$

$$-4D_3(2w,_{xss} + v,_{xs}/R)]_0^L = 0$$

VIII. DEVELOPMENT OF A DONNELL TYPE DIFFERENTIAL EQUATION FOR THE STATIC CASE

The stability equilibrium equations are written in the following form:

$$u_{,xs} = ap_1 v + a_4 v_{,xx} + a_2 v_{,ss} + a_5 w_{,xxs} + ap_2 w_{,s} \tag{16a}$$

$$v_{,xs} = a_1 u_{,xx} + a_3 u_{,ss} + a_6 w_{,x} \tag{17a}$$

$$c_1 u_{,x} + cp_1 v_{,s} + c_2 v_{,xxs} = b_4 w + [bp_2 - M_a R \cos(s/R)]w_{,xx} + bp_1 w_{,ss}$$

$$+ b_1 w_{,xxxx} + b_3 w_{,xxss} + b_2 w_{,ssss} \tag{18a}$$

where:

$$ap_1 = P_o a_7$$

$$ap_2 = P_o a_8 + a_9$$

$$bp_1 = P_o b_5 + b_6 \tag{20}$$

$$bp_2 = P_o b_7 + P_a b_7 + b_8$$

$$cp_1 = P_o c_3 + c_4$$

and:

$$a_1 = -2a_1/(\mu_{xs} a_2 + 2a_3)$$

$$a_2 = -2a_2/(\mu_{xs} a_2 + 2a_3)$$

$$a_3 = -2a_3/(\mu_{xs} a_2 + 2a_3)$$

$$a_4 = -2(D_3 + a_3 R^2)/R^2(\mu_{xs} a_2 + 2a_3) \tag{20 a}$$

$$a_5 = -4D_3/R(\mu_{xs} a_2 + 2a_3)$$

$$a_6 = a_2 \mu_{xs}/R(\mu_{xs} a_2 + 2a_3)$$

$$a_7 = 1/R(\mu_{xs} a_2 + 2a_3)$$

$$a_8 = 1/(\mu_{xs} a_2 + 2a_3)$$

$$a_9 = -2a_2/R(\mu_{xs} a_2 + 2a_3)$$

$$b_1 = 2D_1 R$$

$$b_2 = 2D_2 R$$

$$b_3 = 4D_1 R \mu_{sx} + 8D_3 R$$

$$b_4 = 2(R^2 a_2 + k^2 D_2)/R^3$$

$$b_5 = -R^2$$

$$b_6 = 4kD_2/R$$

$$b_7 = -R^2/2$$

$$b_8 = 4\mu_{sx} kD_1/R$$

$$c_1 = a_2 \mu_{xs}$$

$$c_2 = -4D_3$$

$$c_3 = R$$

$$c_4 = 2a_2$$

(20a Con't.)

A linear differential operation is defined as follows:

$$Q = a_1 ap_1 \frac{\partial^2}{\partial x^2} + a_3 ap_1 \frac{\partial^2}{\partial s^2} + a_1 a_4 \frac{\partial^4}{\partial x^4} + (a_1 a_2 + a_3 a_4 - 1) \frac{\partial^4}{\partial x^2 \partial s^2} + a_2 a_3 \frac{\partial^4}{\partial s^4} \qquad (21)$$

By successive differentiation and combination, Equations (16a) and (17a) will have the following form:

$$Qu = - [a_6 ap_1 w,_x + (a_2 a_6 + ap_2) w,_{xss} + a_4 a_6 w,_{xxx} + a_5 w,_{xxxss}] \qquad (22a)$$

$$Q_v = - [(a_1 ap_2 + a_6) w,_{xxs} + a_3 ap_2 w,_{sss} + a_1 a_5 w,_{xxxxs} + a_3 a_5 w,_{xxsss}] \qquad (22b)$$

Operating on Equating (18a) with the differential operator defined in Equation (21) results in the following:

$$Q(c_1 u,_x + cp_1 v,_s + c_2 v,_{xxs})$$

$$= Q \ [b_4 w + bp_2 w,_{xx} - M_a R \cos(s/R) w,_{xx} + bp_1 w,_{ss} + b_1 w,_{xxxx} \qquad (18b)$$

$$+ b_3 w,_{xxss} + b_2 w,_{ssss}]$$

By utilizing Equations (22a) and (22b), all the u and v terms in Equation (18b) can be eliminated. The resulting equation, an eighth order differential equation in w, is the required Donnell-type differential equation and is given as follows:

$$
\begin{aligned}
& h_{80} w_{,xxxxxxxx} + h_{62} w_{,xxxxxxss} + h_{44} w_{,xxxxssss} + h_{26} w_{,xxssssss} \\
& + h_{08} w_{,ssssssss} + [h_{60} + h_{c60} M_a \cos(s/R)] \, w_{,xxxxxx} + [h_{42} + h_{c42} M_a \cos(s/R)] w_{,xxxxss} \\
& + [h_{24} + h_{c24} M_a \cos(s/R)] \, w_{,xxssss} + h_{06} w_{,ssssss} + h_{s41} M_a \sin(s/R) \, w_{,xxxxs} \\
& + h_{s23} M_a \sin(s/R) \, w_{,xxsss} + [h_{40} + h_{c40} M_a \cos(s/R)] \, w_{,xxxx} + [h_{22} + h_{c22} M_a \cos(s/R)] w_{,xxss} \\
& + h_{04} \, w_{,ssss} + h_{s21} M_a \sin(s/R) \, w_{,xxs} + [h_{20} + h_{c20} M_a \cos(s/R)] \, w_{,xx} + h_{02} w_{,ss} = 0
\end{aligned}
\tag{23}
$$

where:

$$h_{80} = d_{80}$$

$$h_{62} = d_{62}$$

$$h_{44} = d_{44}$$

$$h_{26} = d_{26}$$

$$h_{08} = d_{08}$$

$$h_{60} = d_{60} + e_{60} P_o + f_{60} P_a$$

$$h_{c60} = g_{60}$$

$$h_{42} = d_{42} + e_{42} P_o + f_{42} P_a$$

$$h_{c42} = g_{42}$$

$$h_{24} = d_{24} + e_{24} P_o + f_{24} P_a$$

$$h_{c24} = g_{24}$$

$$\tag{24}$$

$$h_{06} = d_{06} + e_{06}P_o$$

$$h_{s41} = \bar{g}_{41}$$

$$h_{s23} = \bar{g}_{23}$$

$$h_{40} = d_{40} + e_{40}P_o + \bar{e}_{40}P_o^2 + ef_{40}P_oP_a$$

$$h_{c40} = g_{40} + eg_{40}P_o$$

$$h_{22} = d_{22} + e_{22}P_o + \bar{e}_{22}P_o^2 + ef_{22}P_oP_a$$

$$h_{c22} = eg_{22}P_o + g_{22}$$

$$h_{04} = d_{04} + e_{04}P_o + \bar{e}_{04}P_o^2$$

$$h_{s21} = e\bar{g}_{21}P_o + \bar{g}_{21}$$

$$h_{20} = e_{20}P_o$$

$$h_{c20} = eg_{20}P_o + g_{20}$$

$$h_{02} = e_{02}P_o$$

(24 Con't.)

and:

$$d_{80} = a_1 a_4 b_1$$

$$d_{62} = a_1 a_5 c_2 + b_1(a_1 a_2 + a_3 a_4 - 1) + a_1 a_4 b_3$$

$$d_{44} = a_3 a_5 c_2 + a_2 a_3 b_1 + b_3(a_1 a_2 + a_3 a_4 - 1) + a_1 a_4 b_2$$

(25)

$$d_{26} = a_2 a_3 b_3 + b_2(a_1 a_2 + a_3 a_4 - 1)$$

$$d_{08} = a_2 a_3 b_2$$

$$d_{60} = a_1 a_4 b_8$$

$$e_{60} = a_1 a_7 b_1 + a_1 a_4 b_7$$

$$f_{60} = a_1 a_4 b_7$$

$$g_{60} = -R a_1 a_4$$

$$d_{42} = a_5 c_1 + a_1 a_5 c_4 + a_1 a_9 c_2 + a_6 c_2 + b_8(a_1 a_2 + a_3 a_4 - 1) + a_1 a_4 b_6$$

$$e_{42} = a_1 a_5 c_3 + a_1 a_8 c_2 + a_1 a_4 b_5 + a_3 a_7 b_1 + a_1 a_7 b_3 + b_7(a_1 a_2 + a_3 a_4 - 1)$$

$$f_{42} = b_7(a_1 a_2 + a_3 a_4 - 1)$$

$$g_{42} = -R(a_1 a_2 + a_3 a_4 - 1)$$

$$d_{24} = a_3 a_5 c_4 + a_3 a_9 c_2 + a_2 a_3 b_8 + b_6(a_1 a_2 + a_3 a_4 - 1)$$

$$e_{24} = a_3 a_5 c_3 + a_3 a_8 c_2 + b_5(a_1 a_2 + a_3 a_4 - 1) + a_3 a_7 b_3 + a_1 a_7 b_2 + a_2 a_3 b_7$$

$$f_{24} = a_2 a_3 b_7 \tag{25 Con't.}$$

$$g_{24} = -R a_2 a_3$$

$$d_{06} = a_2 a_3 b_6$$

$$e_{06} = a_2 a_3 b_5 + a_3 a_7 b_2$$

$$\bar{g}_{41} = 2(a_1 a_2 + a_3 a_4 - 1)$$

$$\bar{g}_{23} = 4 a_2 a_3$$

$$d_{40} = a_4 a_6 c_1 + a_1 a_4 b_4$$

$$e_{40} = a_1 a_7 b_8$$

$$ef_{40} = a_1 a_7 b_7$$

$$g_{40} = (1/R)(a_1 a_2 + a_3 a_4 - 1)$$

$$eg_{40} = -Ra_1a_7$$

$$\bar{e}_{40} = a_1a_7b_7$$

$$d_{22} = a_2a_6c_1 + a_1a_9c_4 + a_6c_4 + b_4(a_1a_2 + a_3a_4 - 1) + a_9c_1$$

$$e_{22} = a_8c_1 + a_1a_9c_3 + a_1a_8c_4 + a_6c_3 + a_3a_7b_8 + a_1a_7b_6$$

$$\bar{e}_{22} = a_1a_8c_3 + a_1a_7b_5 + a_3a_7b_7$$

$$ef_{22} = a_3a_7b_7$$

$$g_{22} = (1/R)(6a_2a_3)$$

$$eg_{22} = a_3a_7R$$

$$d_{04} = a_3a_9c_4 + a_2a_3b_4$$

$$e_{04} = a_3a_9c_3 + a_3a_8c_4 + a_3a_7b_6$$

$$\bar{e}_{04} = a_3a_8c_3 + a_3a_7b_5$$

$$e\bar{g}_{21} = 2a_3a_7$$

$$\bar{g}_{21} = (1/R^2)(-4a_2a_3)$$

$$e_{20} = a_6a_7c_1 + a_1a_7b_4$$

$$eg_{20} = (1/R)\, a_3a_7$$

$$g_{20} = (1/R^3)(-a_2a_3)$$

$$e_{02} = a_3a_7b_4$$

IX. DETERMINATION OF THE CRITICAL RESULTANT END MOMENT BY USE OF THE RITZ METHOD

The classical Ritz method solution for the static buckling of a cylindrical shell, as shown in Reference (1), requires that an assumption be made for the shape of the buckled cylinder. The following expression is assumed for the radial deflection:

$$w = [A_1 \cos (s/R) + A_2 \cos (2s/R) + A_3 \cos (3s/R) + A_4 \cos (4s/R) + A_5 \cos (5s/R)$$

$$+ A_6 \cos (6s/R)] \ [\sin(m\pi x/L)] \tag{26}$$

where A_1 through A_6 are arbitrary displacement parameters; and m is an arbitrary positive interger representing the number of buckling modes in the axial direction. The assumed radial deflection expression satisfies the boundary condition for the coordinates x and s. These boundary conditions, for a simply supported shell, are zero deflection and moment at the ends of the cylinder and a periodicity of 2π, respectively, for the x and s coordinates. The boundary conditions represented mathematically are:

$$w(x,s) = w(0,s) = w(L,s) = 0 \tag{27a}$$

$$w,_{xx}(x,s) = w,_{xx}(0,s) = w,_{xx}(L,s) = 0 \tag{27b}$$

for the x coordinate, and:

$$w(x,s) = w(x,s + 2\pi) \tag{27c}$$

for the s coordinate.

For convenience, Equation (26) will be written in the following summation form:

$$w = [A_n \cos (ns/R)] \ [\sin (m\pi x/L)] \tag{26a}$$

where the n, an interger, is the summing index and has the values 1 through 6.

Substitution of Equation (26a) into Equation (23) will result in a residual force per unit area F, and Equation (23) can be written in the following form:

$$F = [\sin (m\pi x/L)] \ [G_1 + G_2 + G_3] \tag{28}$$

where

$$G_1 = M_a A_n B_{1n} \cos(s/R) \cos(ns/R)$$

$$G_2 = M_a A_n B_{2n} \sin(s/R) \sin(ns/R) \tag{29}$$

$$G_3 = A_n B_{3n} \cos(ns/R)$$

and

$$B_{1n} = R^3 [-h_{c60}(\lambda^6/R^6) - h_{c42}(\lambda^4 n^2/R^6) - h_{c24}(\lambda^2 n^4/R^6) + h_{c40}(\lambda^4/R^4)$$

$$+ h_{c22}(\lambda^2 n^2/R^4) - h_{c20}(\lambda^2/R^2)]$$

$$B_{2n} = R^3 [-h_{s41}(\lambda^4 n/R^5) - h_{s23}(\lambda^2 n^3/R^5) + h_{s21}(\lambda^2 n/R^3)] \tag{30}$$

$$B_{3n} = R^3 [h_{80}(\lambda^8/R^8) + h_{62}(\lambda^6 n^2/R^8) + h_{44}(\lambda^4 n^4/R^8) + h_{26}(\lambda^2 n^6/R^8)$$

$$+ h_{08}(n^8/R^8) - h_{60}(\lambda^6/R^6) - h_{42}(\lambda^4 n^2/R^6) - h_{24}(\lambda^2 n^4/R^6)$$

$$- h_{06}(n^6/R^6) + h_{40}(\lambda^4/R^4) + h_{22}(\lambda^2 n^2/R^4) + h_{04}(n^4/R^4)$$

$$- h_{20}(\lambda^2/R^2) - h_{02}(n^2/R^2)]$$

$$\lambda = m\pi R/L$$

Equation (26a) can be written again with a summation notation but using a different index, in the following form:

$$w = [A_r \cos(rs/R)] [\sin(m\pi x/L)] \tag{26b}$$

where the r, an interger, is the summing index and has the values 1 through 6. Equation (26b) can now be written in the form:

$$w = G_o \sin(m\pi x/L) \tag{31}$$

where

$$G_o = A_r \cos(rs/R) \tag{32}$$

The work done by the residual force, F, during the radial deflection of buckling is obtained from the product of Equations (28) and (31) and is given as follows:

$$Fw = [\sin^2 (m\pi x/L)] \; [G_o G_1 + G_o G_2 + G_o G_3] \qquad (33)$$

The expression for Fw must be evaluated over the surface area of the cylindrical shell to obtain the total work W expression given as follows:

$$W = \int_0^L \int_0^{2\pi R} Fw \; ds \; dx \qquad (34)$$

Substitution of Equation (33) into (34) will result in the following expression:

$$W = \int_0^L \int_0^{2\pi R} [\sin^2 (m\pi x/L)] \; [G_o G_1 + G_o G_2 + G_o G_3] \; ds \; dx \qquad (35)$$

Integrating Equation (35) with respect to x will give the following:

$$W = (L/2) \int_0^{2\pi R} [G_o G_1 + G_o G_2 + G_o G_3] \; ds \qquad (36)$$

Evaluation of the integrals of the products $G_o G_1$, $G_o G_2$, and $G_o G_3$ will give the following:

$$(L/2) \int_0^{2\pi R} G_o G_1 \; ds = M_a B_{1n} A_n A_r (L/2) \int_0^{2\pi R} \cos (s/R) \cos (ns/R) \cos (rs/R) \; ds$$

$$= M_a B_{1n} A_n A_r (\pi RL/4), \text{ when } r = n \pm 1 \qquad (37a)$$

$$(L/2) \int_0^{2\pi R} G_o G_2 \; ds = M_a B_{2n} A_n A_r (L/2) \int_0^{2\pi R} \sin (s/R) (ns/R) \cos (rs/R) \; ds$$

$$= M_a B_{2n} A_n A_r (\pi RL/4), \text{ when } r = n - 1 \; ; \qquad (37b)$$

$$\text{and } = -M_a B_{2n} A_n A_r (\pi RL/4), \text{ when } r = n+1$$

$$(L/2) \int_0^{2\pi R} G_o G_3 \, ds = B_{3n} A_n A_r (L/2) \int_0^{2\pi R} \cos(ns/R) \cos(rs/R) \, ds$$

$$= B_{3n} A_n A_r (\pi RL/4) \text{ when } r = n \tag{37c}$$

All other combinations of r and n values not specified by the r-n condition equations will cause the integrals to vanish. Substitution of the specified values for r and n into Equations (37a), (37b) and (37c); evaluation of these equations; and separation of terms will result in the following general expression for the total work during the buckling due to the radial deflection:

$$W = (\pi RL/4) \left[\sum_{n=1}^{6} A_n^2 B_{3n} + M_a \sum_{n=1}^{5} A_n A_{n+1} (B_{1,n} + B_{1,n+1} - B_{2,n} + B_{2,n+1}) \right] \tag{38}$$

where the values of n are as specified on the summation symbols.

Equation (38) is minimized with respect to the arbitrary displacement parameters A_n when n again has the interger values 1 through 6. This procedure will result in the following system of algebraic equations:

$$\frac{\partial W}{\partial A_1} = 0 : \quad A_1 \bar{B}_{11} + A_2^{M_a} \bar{B}_{12} = 0 \tag{39}$$

$$\frac{\partial W}{\partial A_2} = 0 : \quad A_1^{M_a} \bar{B}_{21} + A_2 \bar{B}_{22} + A_3^{M_a} \bar{B}_{23} = 0 \tag{40}$$

$$\frac{\partial W}{\partial A_3} = 0 : \quad A_2^{M_a} \bar{B}_{32} + A_3 \bar{B}_{33} + A_4^{M_a} \bar{B}_{34} = 0 \tag{41}$$

$$\frac{\partial W}{\partial A_4} = 0 : \quad A_3^{M_a} \bar{B}_{43} + A_4 \bar{B}_{44} + A_5^{M_a} \bar{B}_{45} = 0 \tag{42}$$

$$\frac{\partial W}{\partial A_5} = 0 : \quad A_4^{M_a} \bar{B}_{54} + A_5 \bar{B}_{55} + A_6^{M_a} \bar{B}_{56} = 0 \tag{43}$$

$$\frac{\partial W}{\partial A_6} = 0 : \quad A_5^{M_a} \bar{B}_{65} + A_6 \bar{B}_{66} = 0 \tag{44}$$

where:

$$\bar{B}_{n,n} = 2B_{3n}; \qquad n = 1 \text{ to } 6 \qquad (45)$$

$$\bar{B}_{n,n+1} = \bar{B}_{n+1,n} = (B_{1,n} + B_{1,n+1} - B_{2,n} + B_{2,n+1}); \quad n = 1 \text{ to } 5$$

The coefficients of the A_n terms in Equations (39) through (44), when written in determinate form, result in the following expression:

$$(\bar{D}) = \begin{vmatrix} \bar{B}_{11} & M_a\bar{B}_{12} & 0 & 0 & 0 & 0 \\ M_a\bar{B}_{21} & \bar{B}_{22} & M_a\bar{B}_{23} & 0 & 0 & 0 \\ 0 & M_a\bar{B}_{32} & \bar{B}_{33} & M_a\bar{B}_{34} & 0 & 0 \\ 0 & 0 & M_a\bar{B}_{43} & \bar{B}_{44} & M_a\bar{B}_{45} & 0 \\ 0 & 0 & 0 & M_a\bar{B}_{54} & \bar{B}_{55} & M_a\bar{B}_{56} \\ 0 & 0 & 0 & 0 & M_a\bar{B}_{65} & \bar{B}_{66} \end{vmatrix} \qquad (46)$$

and Equations (39) through (44) can be written in the following matrix form:

$$[\bar{D}] \quad [A_n] \quad = 0 \qquad (47)$$

Since the arbitrary displacement parameters, A_n, are real; the determinant, $(\bar{D})$, must vanish for all values of A_n. Therefore, evaluation of the determinant $(\bar{D})$, which results in a sixth degree equation in M_a, will give the critical resultant moment, M_a cr, of the circular cylinder for the particular values of p_o and P_a used in the evaluation of the h-constants in Equation (24). The desired value of M_a cr is the lowest, positive, real root of the following characteristic equation:

$$T_o + T_1 M_a^2 + T_2 M_a^4 + T_3 M_a^6 = 0 \qquad (48)$$

where:

$$T_o = [\bar{B}_{11} \bar{B}_{22} \bar{B}_{33} \bar{B}_{44} \bar{B}_{55} \bar{B}_{66}]$$

$$T_1 = - [\bar{B}_{11} \bar{B}_{22} \bar{B}_{33} \bar{B}_{44} \bar{B}_{56}^2 + \bar{B}_{11} \bar{B}_{22} \bar{B}_{33} \bar{B}_{66} \bar{B}_{45}^2$$
$$+ \bar{B}_{11} \bar{B}_{22} \bar{B}_{55} \bar{B}_{66} \bar{B}_{34}^2 + \bar{B}_{11} \bar{B}_{44} \bar{B}_{55} \bar{B}_{66} \bar{B}_{23}^2$$
$$+ \bar{B}_{33} \bar{B}_{44} \bar{B}_{55} \bar{B}_{66} \bar{B}_{12}^2]$$

$$T_2 = [\bar{B}_{11} \bar{B}_{22} \bar{B}_{34}^2 \bar{B}_{56}^2 + \bar{B}_{11} \bar{B}_{44} \bar{B}_{23}^2 \bar{B}_{56}^2 + \bar{B}_{11} \bar{B}_{66} \bar{B}_{23}^2 \bar{B}_{45}^2$$
$$+ \bar{B}_{33} \bar{B}_{44} \bar{B}_{12}^2 \bar{B}_{56}^2 + \bar{B}_{33} \bar{B}_{66} \bar{B}_{12}^2 \bar{B}_{45}^2 + \bar{B}_{55} \bar{B}_{66} \bar{B}_{12}^2 \bar{B}_{34}^2]$$

$$T_3 = - [\bar{B}_{12}^2 \bar{B}_{34}^2 \bar{B}_{56}^2]$$

(49)

X. CONCLUSIONS

The general instability of an orthotropic circular cylinder subjected to an axial load, end moment, and uniform radial pressure has been analyzed by a technique paralleling the technique used by Bodner (1). The analysis has been successfully programmed, see Appendix C, and the program has been run with arbitrary data. The results obtained with the arbitrary data could only be visually checked and were within the range of expected results. The program has not been used in conjunction with experimental investigations.

XI. RECOMMENDATIONS

It is assumed that this investigation of orthotropic shells will be continued on an experimental basis, and that the experiments will attempt to verify and/or modify the existing analysis as well as refine and modify the computer program that has been written. The recommendations that are stated are intended as a guide for the experimental investigators.

The deflection expression, Equation 26, should be extended to a minimum of 12 circumferential deflection terms and possibly extended to 16 or 24 terms should computer capacity allow this extension. This extension will improve the accuracy of the analysis.

The axial term of the deflection expression, the sine term, should be extended to contain a cosine term, that is, $\sin(m\pi x/L) + \cos(m\pi x/L)$. The axial term will then allow a variation of end conditions, which become significant in the short cylinder range

and possibly the intermediate cylinder range. This modified axial term can also be used to induce deflections due to the pre-buckling stresses.

An additional term can be added to the deflection expression to account for the initial imperfections of the cylinder.

The discarded roots of the characteristic equation should be mathematically investigated, and the meaning of the imaginary roots should be ascertained.

The sensitivity of the program should be checked for each of the dependant variables, geometric and loading. Each modification of the program should be checked for the possible changes in sensitivity that can be expected.

A normalization of the final program is recommended which will allow a comparison with other information existing in the field.

Since stability of orthotropic shells is both a general and local stability problem the program can be extended to include the local stability problem by evaluating existing investigations in this field.

Results obtained by other investigators can be checked with the program to determine whether or not the program is valid.

e_{xx}, e_{ss}, e_{xs} — Axial, circumferential, and shear strains, respectively, occurring during the buckling process, defined by Equation 1.

e_{60}, e_{42}, etc. — Constants for the Donnell differential equation defined by Equation 25.

$\bar{e}_{40}$, $\bar{e}_{22}$, $\bar{e}_{04}$ — Constants for the Donnell differential equation defined by Equation 25.

ef_{40}, ef_{22} — Constants for the Donnell differential equation defined by Equation 25.

eg_{40}, eg_{22}, eg_{20} — Constants for the Donnell differential equation defined by Equation 25.

$\bar{eg}_{21}$ — Constants for the Donnell differential equation defined by Equation 25.

F — Residual force per unit area remaining in the shell as a result of the assumed deflection expression.

f_{60}, f_{42}, f_{24} — Constants for the Donnell differential equation defined by Equation 25.

G — Average shear modulus, where $G = E/(1 + \mu)$

G_0 — Constant defined by Equation 32.

G_1, G_2, G_3 — Constants for the residual force equation defined by Equation 29.

g_{60}, g_{42}, etc. — Constants for the Donnell differential equation defined by Equation 25.

$\bar{g}_{41}$, $\bar{g}_{23}$, $\bar{g}_{21}$ — Constants for the Donnell differential equation defined by Equation 25.

h — Shell wall thickness.

h_{eq} — Shell wall thickness modified for the orthotropic case.

h_{80}, h_{60}, etc. — Constants for the Donnell differential equation defined by Equation 24.

h_{c60}, h_{c42}, etc. — Constants for the Donnell differential equation defined by Equation 24.

h_{s41}, h_{s23}, etc.   – Constants for the Donnell differential equation defined by Equation 24.

k   – Non-dimensional integer constant.

L   – Length of cylindrical shell.

M_a – Modified end moment defined by $M_a = M_o / \pi R^2$

M_o – Applied end moment.

m   – Number of buckling modes in the axial direction.

$\overline{N}_{xx}$, $\overline{N}_{ss}$, $\overline{N}_{xs}$ – Axial, circumferential, and shear stress resultants in the shell just prior to buckling defined by Equation 5.

P_a – Modified axial load defined by $P_a = P_o / \pi R^2$.

P_o – Applied axial load.

P_o – Applied uniform radial pressure.

$\pi R / L$ – Circular shell radius to length ratio.

Q   – Linear differential operator defined by Equation 21.

R   – Radius of circular shell.

R/h   – Circular shell radius to thickness ratio.

s   – Circumferential coordinate of circular shell.

T_o, T_1, etc.   – Constants for the characteristic equation defined by Equation 49.

U   – Change in strain energy during buckling.

u   – Axial deformation of an element of the circular shell.

V   – Change in potential energy during buckling.

V_s – Volume of circular shell wall.

v   – Circumferential deformation of an element of the circular shell.

W   — Total work due to the residual force during buckling.

w   — Radial deformation of an element of the circular shell.

x   — Axial coordinate of the circular shell.

z   — Radial coordinate of the circular shell.

a_1, a_2, a_3 — Extensional stiffnesses for the axial, circumferential, and shear strains respectively.

δ — Variational symbol.

θ — Coordinate angle corresponding to the circumferential coordinate, where $\theta = s/R$.

λ — Constant defined by Equation 30.

μ — Poison's ratio for the isotropic case.

μ_{xs}, μ_{sx} — Poison's ratios from the x to s and the s to x directions, respectively.

σ_{xx}, σ_{ss}, σ_{xs} — Axial, circumferential, and shear stresses, respectively, occurring during the buckling process.

$\bar{\sigma}_{xx}$, $\bar{\sigma}_{ss}$, $\bar{\sigma}_{xs}$ — Axial, circumferential, and shear stresses, respectively, in the circular shell just prior to buckling.

APPENDIX B

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APPENDIX C

COMPUTER PROGRAM

The solution of the problem being investigated here requires that an n-th degree polynomial in M_a be solved for the lowest, positive, real or zero root. The degree of this polynomial, the characteristic equation of the stability determinant, Equation 48, is equal to the maximum value used for n in the deflection expression, Equation 26a. The applied end moment can be plus or minus and still have the same stability condition, there-fore the characteristic equation can be considered as a polynomial in M_a^2 and the roots of the characteristic equation are determined by the cubic formula. The values of M_a are then obtained by taking the square root of the M_a^2 value.

In the development of this problem, the total change in energy expression, Equation 11, is manipulated by certain mathematical operations. After each manipulation a new set of constants is obtained. These new constants are defined in terms of previously defined constants, etc., and finally all constants are defined in terms of the extensional stiffnesses and bending rigidities, Equation 3, and other input variables. Therefore, the problem that the computer program must solve is an evaluation of successive sets of constants, and the solution of the characteristic equation for the desired root. A computer program type-out is shown in Appendix D, and this program is written in Fortran II for an IBM 1620 computer.

In the investigation of an orthotropic shell, the a_1, a_2, D_1 and D_2 values, Equation 3, are calculated for a particular orthotropic shell using h_{eq}. These values are then rationed to the respective isotropic shell values, a and D, which are obtained by using h values. These ratios are used as input variables in the form: A1A, A2A, D1D, and D2D; where A1A = a_1/a, etc. Similarly the input values of L and h appear in the computer program as ratios in the form $\pi R/L$ and R/h, respectively.

In any stability problem it is necessary that the sensitivity of any or all variables be investigated, and that a study of the output variable M_a for certain ranges of the input variables be made. An iterative process that incruments the input variables between certain desired limits permits these studies. All input variables can be iterated with the exception of E, μ_{xs}, μ_{sx} and R.

The iterative process requires three input values for each of the following input terms: A1A, A2A, D1D, D2D, R/h, $\pi R/L$, p_o, P_o, k and m. These values are: the initial value, also the minimum; the maximum value; and the increment by which the input variable varies between the initial and maximum values.

The program output is M_a vs. P_o. A sample output format is shown in Appendix E. This sample output format is for arbitrary values of the input variables.

When a constant value, non-incrumented value, of an input variable is used in a particular computer run, the initial value and maximum value must be the same, and the incrument should be an arbitrary positive number.

An increase in the number of terms in the deflection expression will require a change in the root solving portion of the program, since the cubic formula will no longer provide a valid solution to the characteristic equation.

The symbols used in the computer program are self-explanatory except symbols B11, B12, and B13 which are the b_1, b_2 and b_3 constants of Equation 20a, respectively. A partial list of definitions and computer program symbols is given in Appendix F.

Certain constants used in the text of this paper do not appear in the computer program. These constants have been incorporated into succeeding constants with the intent of conserving computer storage.

The program must be precompiled with format, since an overload condition exists on a 40K bit storage when the program is precompiled without format.

APPENDIX D

COMPUTER PROGRAM TYPE-OUT

```
C       PROGRAM FOR THE STABILITY ANALYSIS OF AN ORTHOTROPIC CIRCULAR
C       SHELL WITH AXIAL LOAD, END MOMENT, AND RADIAL PRESSURE.
C       THIS PROGRAM IS WRITTEN IN FORTRAN II FOR AN IBM 1620 COMPUTER.
C       INPUT DATA (5 CARDS) - ALL DATA IN 8 DIGIT FIELDS
  100 READ501,E,VXS,VSX,R
C       VALUES FOR PRESSURE AND AXIAL LOAD
      READ501,APO,POINC,POMAX,ABGPO,BGPOI,BGPOM
C       INITIAL VALUES (MINIMUM)
      READ501,AROH,ARPL,AA1A,AA2A,AD1D,AD2D,AFK,AEM
C       INCRUMENT VALUES
      READ501,ROHIN,RPLIN,A1AIN,A2AIN,D1DIN,D2DIN,EKINC,EMINC
C       MAXIMUM VALUES
      READ501,ROHMX,RPLMX,A1AMX,A2AMX,D1DMX,D2DMX,EKMAX,EMMAX
      DIMENSION V(8),      U(8),S(6,8),B1(8),B2(8),B3(8),X(6),Y(5)
C       REPEATING CONSTANTS
      PI=3.141593
      F1=1.
      F2=2.
      F3=3.
      F4=4.
      PAINC=BGPOI/(PI*R*R)
      PAMAX=BGPOM/(PI*R**2)
C       INITIALIZING STATEMENT
      EK=AEK
C       INITIALIZING STATEMENT
   10 EM=AEM
      U(1)=F1/R
      DO 222 N=2,8
  222 U(N)=U(1)**N
      DO 225 I=1,6
      DO 225 J=1,8
      D=I
  225 S(I,J)=D**J
C       INITIALIZING STATEMENT
   15 RPL=ARPL
C       OUTPUT STATEMENT
      PUNCH510,E,VXS,VSX,R,EK,EM
   25 EL=R*PI/RPL
      V(1)=EM*PI*R/EL
      DO 223 N=2,8
  223 V(N)=V(1)**N
      ROH=AROH
C       OUTPUT STATEMENT
      PUNCH512,RPL
C       INITIALIZING STATEMENT
   35 D2D=AD2D
C       OUTPUT STATEMENT
      PUNCH515,ROH
C       INITIALIZING STATEMENT
   45 D1D=AD1D
C       INITIALIZING STATEMENT
```

```
   50 A2A=AA2A
C     INITIALIZING STATEMENT
   55 A1A=AA1A
C     INITIALIZING STATEMENT
      PO=APO
C     OUTPUT STATEMENT
   60 PUNCH511,A1A,A2A,D1D,D2D
      H=R/ROH
C     CONSTANTS FOR EQULIBRIUM EQUATIONS
      AX=F1-VXS*VSX
      Q1=F1/(E*H*(VSX/(F2*AX)-F1/(F4*(F1-VXS))))
      A1=-F2*Q1*A1A*E*H/(F2*AX)
      A2=A1*A2A/A1A
      A3=-F2*Q1*E*H/(8.*(F1+VXS))
      A5=-F4*(Q1/R)*E*H**F3/(96.*(F1+VXS))
      A4=A3+A5/(F2*R)
      A6=A2*VXS/(-F2*R)
      A7=Q1/R
      A9=-F2*A6/VXS
      B11=F2*R*D1D*E*H**F3/(24.*AX)
      B12=B11*D2D/D1D
      B13=E*H**F3*(F2*D1D*VSX/AX+F1/(F1+VSX))*R/12.
      B4=-A2*R/Q1+B12*EK*EK/R**F4
      B5=-R*R
      B6=B12*F2*EK/(R*R)
      B7=-(R**F2)/F2
      B8=B11*F2*VSX*EK/(R*R)
      C1=VSX*A2A*E*H/(F2*AX)
      C =-E*H**F3/(24.*(F1+VXS))
      C4=F2*C1/VXS
C     CONSTANTS FOR DONNELL EQUATION (EQUA. 24 AND 25)
      H80=A1*A4*B11
      Q2=A1*A2+A3*A4-F1
      H62=A1*(A5*C2+A4*B13)+B11*Q2
      H44=A3*(A5*C2+A2*B11)+B13*Q2+A1*A4*B12
      H26=A2*A3*B13+B12*Q2
      H08=A2*A3*B12
      E60=A1*(A7*B11+A4*B7)
      HC60=-R*A1*A4
      D42=A1*(A5*C4+A9*C2+A4*B6)+A5*C1+A6*C2+B8*Q2
      E42=A1*(A5*R+Q1*C2+A4*B5+A7*B13)+A3*A7*B11+B7*Q2
      HC42=-R*Q2
      D24=A3*(A5*C4+A9*C2+A2*B8)+B6*Q2
      E24=A3*(A5*R+Q1*C2+A7*B13+A2*B7)+B5*Q2+A1*A7*B12
      HC24=-R*A2*A3
      E06=A3*(A2*B5+A7*B12)
      HS41=F2*Q2
      HS23=F4*A2*A3
      D40=A4*(A6*C1+A1*B4)
      D22=C1*(A2*A6+A9)+C4*(A1*A9+A6)+B4*Q2
      E22=A1*(A9*R+Q1*C4+A7*B6)+A6*R+Q1*C1+A3*A7*B8
      EB22=A1*(Q1*R+A7*B5)+A3*A7*B7
      D04=A3*(A9*C4+A2*B4)
      E04=A3*(A9*R+Q1*C4+A7*B6)
```

```
      EB04=A3*(Q1*R+A7*B5)
      H20=PO*A7*(A6*C1+A1*B4)
C     OUTPUT STATEMENT
  600 PUNCH514,PO
C     INITIALIZING STATEMENT
      BIGPO=ABGPO
      PA=BIGPO/(PI*R*R)
  601 H60=A1*A4*(B8+B7*PA)+E60*PO
      H42=D42+E42*PO+B7*Q2*PA
      H24=D24+E24*PO+A2*A3*B7*PA
      H06=A2*A3*B6+E06*PO
      H40=D40+A1*A7*PO*(B8+B7*(PO+PA))
      HC40=Q2/R-R*A1*A7*PO
      H22=D22+PO*(E22+EB22*PO+A3*A7*B7*PA)
      HC22=A3*(A7*R*PO+6.*A2/R)
      H04=D04+PO*(E04+EB04*PO)
      HS21=F2*A3*(A7*PO-F2*A2/R**F2)
      HC20=A3*(PO*A7/R-A2/R**F3)
      H02=A3*A7*B4*PO
C     STABILITY DETERMINANT CONSTANTS (EQUA.45)
      DO 227 N=1,6
      B1(N)=-HC60*V(6)*U(3)-HC42*V(4)*U(3)*S(N,2)-HC24*V(2)*U(3)*S(N,4)
      B1(N)=B1(N)+HC40*V(4)*U(1)+HC22*V(2)*U(1)*S(N,2)-HC20*V(2)*R
      B2(N)=HS21*V(2)*S(N,1)-HS41*V(4)*U(2)*S(N,1)-HS23*V(2)*U(2)*S(N,3)
      B3(N)=H80*V(8)*U(5)+H62*V(6)*U(5)*S(N,2)+H44*V(4)*U(5)*S(N,4)
      B3(N)=B3(N)+H26*V(2)*U(5)*S(N,6)+H08*U(5)*S(N,8)-H60*V(6)*U(3)
      B3(N)=B3(N)+H42*V(4)*U(3)*S(N,2)-H24*V(2)*U(3)*S(N,4)
      B3(N)=B3(N)-H06*U(3)*S(N,6)+H40*V(4)/R+H22*V(2)*S(N,2)/R
      B3(N)=B3(N)+H04*S(N,4)/R-H02*R*S(N,2)-H20*V(2)*R
C     CONSTANTS FOR CHARACTERISTIC EQUATION (EQUA. 49)
  227 X(N)=F2*B3(N)
      DO 228 N=1,5
  228 Y(N)=(B1(N)+B1(N+1)-B2(N)+B2(N+1))**2
      TO=X(1)*X(2)*X(3)*X(4)*X(5)*X(6)
      T1=-X(1)*X(2)*(X(3)*X(4)*Y(5)+X(3)*X(6)*Y(4)+X(5)*X(6)*Y(3))
      T1=T1-X(4)*X(5)*X(6)*(X(1)*Y(2)+X(3)*Y(1))
      T2=X(1)*(Y(5)*(X(2)*Y(3)+X(4)*Y(2))+X(6)*Y(2)*Y(4))
      T2=T2+Y(1)*(X(3)*(X(4)*Y(5)+X(6)*Y(4))+X(5)*X(6)*Y(3))
      T3=-Y(1)*Y(3)*Y(5)
C     SOLUTION OF CHARACTERISTIC EQUATION (EQUA. 48)
      Q=(F3*T1/T3-(T2/T3)**2 )/F3
      T=(F2*(T2/T3)**3 -9.*T1*T2/T3**2 +27.*TO/T3)/27.
      Z=T**2 /F4+Q**3 /27.
      IF(Z)250,260,270
  270 BIGA=(Z**.5-T/F2)**(F1/F3)
      BIGB=(-(Z**.5)-T/F2)**(F1/F3)
      EMA2=BIGA+BIGB
      IF(EMA2)390,400,400
  260 BIGA=(-T/F2)**(F1/F3)
      EMA21=F2*BIGA
      EMA22=-BIGA
      IF(EMA21)261,262,262
  261 EMA2=EMA22
      GO TO 400
```

```
262 EMA2=EMA21
    GO TO 400
250 THETA=ATANF((-(Q**3 /27.)-(T**2 /F4))**.5/(-T/F2))
    Q3=F2*(-Q/F3)**.5
    EMA21=Q3*COSF(THETA/F3)
    EMA22=Q3*COSF(THETA/F3+F2*PI/F3)
    EMA23=Q3*COSF(THETA/F3+F4*PI/F3)
    IF(EMA21)251,252,252
251 IF(EMA22)258,259,259
258 IF(EMA23)390,282,282
282 EMA2=EMA23
    GO TO 400
259 IF(EMA22-EMA23)280,280,284
284 IF(EMA23)280,281,281
280 EMA2=EMA22
    GO TO 400
281 EMA2=EMA23
    GO TO 400
252 IF(EMA21-EMA22)253,253,285
285 IF(EMA22)253,254,254
253 EEMA2=EMA21
    GO TO 255
254 EEMA2=EMA22
255 IF(EEMA2-EMA23)256,256,286
286 IF(EMA23)256,257,257
257 EMA2=EMA23
    GO TO 400
256 EMA2=EEMA2
400 EMA=EMA2**.5
    EMO=EMA*PI*R**F2
    PUNCH513,BIGPO,EMO
    GO TO 391
390 PUNCH516,BIGPO
C   BEGIN CYCLING OF INPUT DATA
391 PA=PA+PAINC
    BIGPO=BIGPO+BGPOI
    IF(PA-PAMAX)601,601,201
201 PO=PO+POINC
    IF(PO-POMAX)600,600,202
202 A1A=A1A+A1AIN
    IF(A1A-A1AMX)60,60,203
203 A2A=A2A+A2AIN
    IF(A2A-A2AMX)55,55,204
204 D1D=D1D+D1DIN
    IF(D1D-D1DMX)50,50,205
205 D2D=D2D+D2DIN
    IF(D2D-D2DMX)45,45,206
206 ROH=ROH+ROHIN
    IF(ROH-ROHMX)35,35,207
207 RPL=RPL+RPLIN
    IF(RPL-RPLMX)25,25,208
208 EM=EM+EMINC
    IF(EM-EMMAX)15,15,209
209 EK=EK+EKINC
```

```
      IF(EK-EKMAX)10,10,210
C     OUTPUT STATEMENT
  210 PRINT 101
  101 FORMAT(13HLOAD NEW DATA)
  501 FORMAT(8F8.0)
  510 FORMAT(20HE,VXS,VSX,RAD,K,M = ,E8.2,2F5.2,F8.2,2F5.1)
  511 FORMAT(6X22HA1/A,A2/A,D1/D,D2/D = ,F6.2,3F7.2)
  512 FORMAT(2X20HPI X RAD / LENGTH = ,F6.3)
  513 FORMAT(10X11HBIGPO,MO = ,E9.2,3XE13.6)
  514 FORMAT(8X11HPRESSURE = ,F8.2)
  515 FORMAT(4X18HRAD / THICKNESS = ,F9.2)
  516 FORMAT(10X11HBIGPO,MO = ,E9.2,6X4HIMAG)
      GO TO 100
      END
```

APPENDIX E

COMPUTER PROGRAM OUTPUT FORMAT

```
         INPUT DATA FOR THE FOLLOWING OUT-PUT FORMAT

30000000.3        .3        10.

15.      15.        30.       0.       5000.     10000.

1200.     6.        1.        1.       1.        1.       1.       1.

400.      2.        1.        1.       1.        1.       1.       1.

1600.     8.        1.        1.       1.        1.       1.       1.



         OUT-PUT FORMAT

E,VXS,VSX,RAD,K,M =   .30E+08   .30   .30    10.00   1.0   1.0
  PI X RAD / LENGTH =   6.000
     RAD / THICKNESS =    1200.00
       A1/A,A2/A,D1/D,D2/D =    1.00    1.00    1.00    1.00
         PRESSURE =      15.00
           BIGPO,MO =    .00E-99      4.145080E+08
           BIGPO,MO =   5.00E+03      4.145072E+08
           BIGPO,MO =   1.00E+04      4.145060E+08
         PRESSURE =      30.00
           BIGPO,MO =    .00E-99      4.145147E+08
           BIGPO,MO =   5.00E+03      4.145134E+08
           BIGPO,MO =   1.00E+04      4.145121E+08
     RAD / THICKNESS =    1600.00
       A1/A,A2/A,D1/D,D2/D =    1.00    1.00    1.00    1.00
         PRESSURE =      15.00
           BIGPO,MO =    .00E-99      3.108863E+08
           BIGPO,MO =   5.00E+03      3.108854E+08
           BIGPO,MO =   1.00E+04      3.108838E+08
         PRESSURE =      30.00
           BIGPO,MO =    .00E-99      3.108928E+08
           BIGPO,MO =   5.00E+03      3.108916E+08
           BIGPO,MO =   1.00E+04      3.108903E+08
  PI X RAD / LENGTH =   8.000
     RAD / THICKNESS =    1200.00
       A1/A,A2/A,D1/D,D2/D =    1.00    1.00    1.00    1.00
         PRESSURE =      15.00
           BIGPO,MO =    .00E-99      2.344654E+08
           BIGPO,MO =   5.00E+03      2.344708E+08
           BIGPO,MO =   1.00E+04      2.344760E+08
         PRESSURE =      30.00
           BIGPO,MO =    .00E-99      2.344727E+08
           BIGPO,MO =   5.00E+03      2.344779E+08
           BIGPO,MO =   1.00E+04      2.344833E+08
```

```
RAD / THICKNESS =    1600.00
  A1/A,A2/A,D1/D,D2/D =    1.00    1.00    1.00    1.00
    PRESSURE =      15.00
      BIGPO,MO =     .00E-99      1.758522E+08
      BIGPO,MO =    5.00E+03      1.758575E+08
      BIGPO,MO =    1.00E+04      1.758626E+08
    PRESSURE =      30.00
      BIGPO,MO =     .00E-99      1.758593E+08
      BIGPO,MO =    5.00E+03      1.758646E+08
      BIGPO,MO =    1.00E+04      1.758698E+08
```

APPENDIX F
PARTIAL LIST OF DEFINITIONS
OF COMPUTER PROGRAM SYMBOLS

E	Modulus of elasticity for isotropic case.
VXS	Poissoin's ratio for the x to s direction.
VSX	Poisson's ratio for the s to x direction.
R	Radius of the shell.
A1A	$= a_1/a$; where $a = Eh/2 (1 - \mu_{xs} \mu_{sx})$
A2A	$= a_2/a$
D1D	$= D_1/D$; where $D = Eh^3/24 (1 - \mu_{xs} \mu_{sx})$
D2D	$= D_2/D$
ROH	$= R/h$
RPL	$= \pi R/L$
EK	$= k$
EM	$= m$
PO	$= p_o$ radial pressure
BIGPO	$= P_o$ axial load
PA	$= P_a = P_o / \pi R^2$
EMO	$= M_o$ end moment
EMA	$= M_o / \pi R^2$
AA1A	– initial value of A1A (minimum)
AA2A	– initial value of A2A (minimum)
AD1D	– initial value of D1D (minimum)
AD2D	– initial value of D2D (minimum)
AROH	– initial value of ROH (minimum)
ARPL	– initial value of RPL (minimum)
AEK	– initial value of EK (minimum)

AEM   – initial value of EM (minimum)

APO   – initial value of PO (minimum)

ABGPO   – initial value of BIGPO (minimum)

A1AMX   – final value of A1A (maximum)

A2AMX   – final value of A2A (maximum)

D1DMX   – final value of D1D (maximum)

D2DMX   – final value of D2D (maximum)

ROHMX   – final value of ROH (maximum)

RPLMX   – final value of RPL (maximum)

EKMAX   – final value of EK (maximum)

EMMAX   – final value of EM (maximum)

POMAX   – final value of PO (maximum)

BGPOM   – final value of BIGPO (maximum)

A1AIN   – incrument of A1A

A2AIN   – incrument of A2A

D1DIN   – incrument of D1D

D2DIN   – incrument of D2D

ROHIN   – incrument of ROH

RPLIN   – incrument of RPL

EKINC   – incrument of EK

EMINC   – incrument of EM

POINC   – incrument of PO

BGPOI   – incrument of BIGPO

EL   Length of shell

H   Thickness of the shell